A New Method of Formulating Finite Difference Equations – Some Reservoir Simulation Examples

H.B. Hales
Brigham Young University, Provo, UT, USA
2007 ACERC Annual Meeting
February 27, 2007

Overview

- Traditional Method
- New Finite Difference Equations
- Results

Traditional Methods

Finite Difference – Divided Difference Approximations for Derivatives:

Forward Difference Formulation:

$$f(x+\Delta x) = f(x) + \frac{df}{dx}\Delta x + \frac{d^2 f}{dx^2}\frac{\Delta x^2}{2!} + \cdots + \frac{d^n f}{dx^n}\frac{\Delta x^n}{n!}$$

Taylor's Series

$$f(x+2\Delta x) = f(x) + \frac{df}{dx}2\Delta x + \frac{d^2 f}{dx^2}\frac{(2\Delta x)^2}{2!} + \cdots + \frac{d^n f}{dx^n}\frac{(2\Delta x)^n}{n!}$$

$$\vdots$$

$$f(x+n\Delta x) = f(x) + \frac{df}{dx}n\Delta x + \frac{d^2 f}{dx^2}\frac{(n\Delta x)^2}{2!} + \cdots + \frac{d^n f}{dx^n}\frac{(n\Delta x)^n}{n!}$$

Solve for

$$\frac{\partial f}{\partial x}\, O(\Delta x^n), \quad \frac{\partial^2 f}{\partial x^2}\, O(\Delta x^{n-1}), \quad \cdots \quad \frac{\partial^n f}{\partial x^n}\, O(\Delta x^1)$$

FIGURE 23.1 From "Numerical Methods of Engineers" by Chapra & Canale

Forward finite-divided-difference formulas: two versions are presented for each derivative. The latter version incorporates more terms of the Taylor series expansion and is, consequently, more accurate.

First Derivative	Error
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{h}$	$O(h)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$	$O(h^2)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$	$O(h)$
$f''(x_i) = \dfrac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$	$O(h^2)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$	$O(h)$
$f'''(x_i) = \dfrac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3}$	$O(h^2)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$	$O(h)$
$f''''(x_i) = \dfrac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4}$	$O(h^2)$

Backward Difference Formulation:

$$f(x-\Delta x) = f(x) - \frac{df}{dx}\Delta x + \frac{d^2 f}{dx^2}\frac{\Delta x^2}{2!} - \cdots + (-1)^n \frac{d^n f}{dx^n}\frac{\Delta x^n}{n!}$$

Taylor's Series

$$f(x-2\Delta x) = f(x) + \frac{df}{dx}2\Delta x - \frac{d^2 f}{dx^2}\frac{(2\Delta x)^2}{2!} - \cdots + (-1)^n \frac{d^n f}{dx^n}\frac{(2\Delta x)^n}{n!}$$

$$\vdots$$

$$f(x-n\Delta x) = f(x) - \frac{df}{dx}n\Delta x + \frac{d^2 f}{dx^2}\frac{(n\Delta x)^2}{2!} - \cdots + (-1)^n \frac{d^n f}{dx^n}\frac{(n\Delta x)^n}{n!}$$

Solve for

$$\frac{\partial f}{\partial x}\, O(\Delta x^n), \quad \frac{\partial^2 f}{\partial x^2}\, O(\Delta x^{n-1}), \quad \cdots \quad \frac{\partial^n f}{\partial x^n}\, O(\Delta x^1)$$

From “Numerical Methods of Engineers” by Chapra & Canale

NUMERICAL DIFFERENTIATION

First Derivative

	Error
$f'(x_i) = \dfrac{f(x_i) - f(x_{i-1})}{h}$	$O(h)$
$f'(x_i) = \dfrac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h}$	$O(h^2)$

Second Derivative

$f''(x_i) = \dfrac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$	$O(h)$
$f''(x_i) = \dfrac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2}$	$O(h^2)$

Third Derivative

$f'''(x_i) = \dfrac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3}$	$O(h)$
$f'''(x_i) = \dfrac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3}$	$O(h^2)$

Fourth Derivative

$f''''(x_i) = \dfrac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4}$	$O(h)$
$f''''(x_i) = \dfrac{3f(x_i) - 14f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4}$	$O(h^2)$

Central Difference Formulation (n even):

Taylor's Series

$$f\left(x+\tfrac{n}{2}\Delta x\right)= f(x)+\frac{df}{dx}\tfrac{n}{2}\Delta x+\frac{d^2 f}{dx^2}\frac{\left(\tfrac{n}{2}\Delta x\right)^2}{2!}+\cdots+\frac{d^n f}{dx^n}\frac{\left(\tfrac{n}{2}\Delta x\right)^n}{n!}$$

$$\vdots$$

$$f(x+2\Delta x)= f(x)+\frac{df}{dx}2\Delta x+\frac{d^2 f}{dx^2}\frac{(2\Delta x)^2}{2!}+\cdots+\frac{d^n f}{dx^n}\frac{(2\Delta x)^n}{n!}$$

$$f(x+\quad\Delta x)= f(x)+\frac{df}{dx}\quad\Delta x+\frac{d^2 f}{dx^2}\quad\frac{\Delta x^2}{2!}+\cdots+^n\frac{d^n f}{dx^n}\quad\frac{\Delta x^n}{n!}$$

$$f(x-\quad\Delta x)= f(x)-\frac{df}{dx}\quad\Delta x+\frac{d^2 f}{dx^2}\quad\frac{\Delta x^2}{2!}-\cdots+(-1)^n\frac{d^n f}{dx^n}\quad\frac{\Delta x^n}{n!}$$

$$f(x-2\Delta x)= f(x)-\frac{df}{dx}2\Delta x-\frac{d^2 f}{dx^2}\frac{(2\Delta x)^2}{2!}-\cdots+(-1)^n\frac{d^n f}{dx^n}\frac{(2\Delta x)^n}{n!}$$

$$\vdots$$

$$f\left(x-\tfrac{n}{2}\Delta x\right)= f(x)-\frac{df}{dx}\tfrac{n}{2}\Delta x+\frac{d^2 f}{dx^2}\frac{\left(\tfrac{n}{2}\Delta x\right)^2}{2!}-\cdots+(-1)^n\frac{d^{\frac{n}{2}} f}{dx^n}\frac{\left(\tfrac{n}{2}\Delta x\right)^{\frac{n}{2}}}{\tfrac{n}{2}!}$$

Solve for $\dfrac{\partial f}{\partial x}\,O(\Delta x^n),\quad \dfrac{\partial^2 f}{\partial x^2}\,O(\Delta x^n),\quad \cdots\quad \dfrac{\partial^n f}{\partial x^n}\,O(\Delta x^2)$

From “Numerical Methods of Engineers” by Chapra & Canale

First Derivative — Error

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \; * \qquad O(h^2)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} \qquad O(h^4)$$

Second Derivative

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \; * \qquad O(h^2)$$

$$f''(x_i) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2} \qquad O(h^4)$$

Third Derivative

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3} \qquad O(h^2)$$

$$f'''(x_i) = \frac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3} \qquad O(h^4)$$

Fourth Derivative

$$f''''(x_i) = \frac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4} \qquad O(h^2)$$

$$f''''(x_i) = \frac{-f(x_{i+3}) + 12f(x_{i+2}) + 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) + f(x_{i-3})}{6h^4} \qquad O(h^4)$$

* Most widely used.

The same FDE's can be derived by assuming a polynomial solution.

- $f(x) = ax^2 + bx + c$

If $f(x_{i-1})$, $f(x_i)$ and $f(x_{i+1})$ are known then

$$\left.\frac{\partial f}{\partial x}\right|_{x_i} = \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}}$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x_i} = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(x_{i+1} - x_i)^2}$$

- The purpose of this work is to investigate finite difference equations based on functions representing the physics of the process.

When don’t Traditional Methods work well?

- Singularities
- Discontinuities
- With strong nonlinearities

For Example:

- Point sources or sinks
- Line sources or sinks
- Multi-phase flow

Reservoir Simulation is such a problem

- Reservoir simulators are computer programs that simulate the flow of oil, gas and water through naturally occurring underground accumulations know as reservoirs.
- Reservoir simulations are run repeatedly in order to optimize hydrocarbon recovery.
- Reservoirs contain wells that cause near singularities in the reservoir pressure.

Lets use functions with singularities to simulate reservoir pressures.

- Around a straight line well p~ln(r)
- Around a point source or sink p~1/r

Try PDE's based on

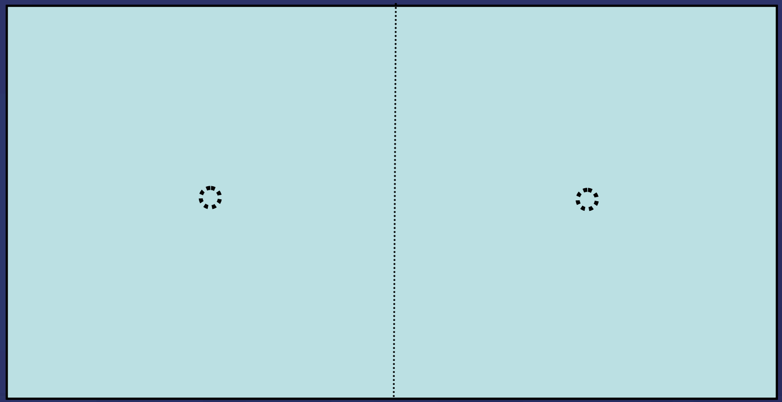
$$P = a\ln(r) + bx + d$$

The resulting FDE’s contain pseudo-permeabilities:

$$K'_x = K_x \frac{(x_{i+1} - x_i)\sum \frac{Qx}{r^2}}{\sum Q \ln(r_{i+1}) - \sum Q \ln(r_i)}$$

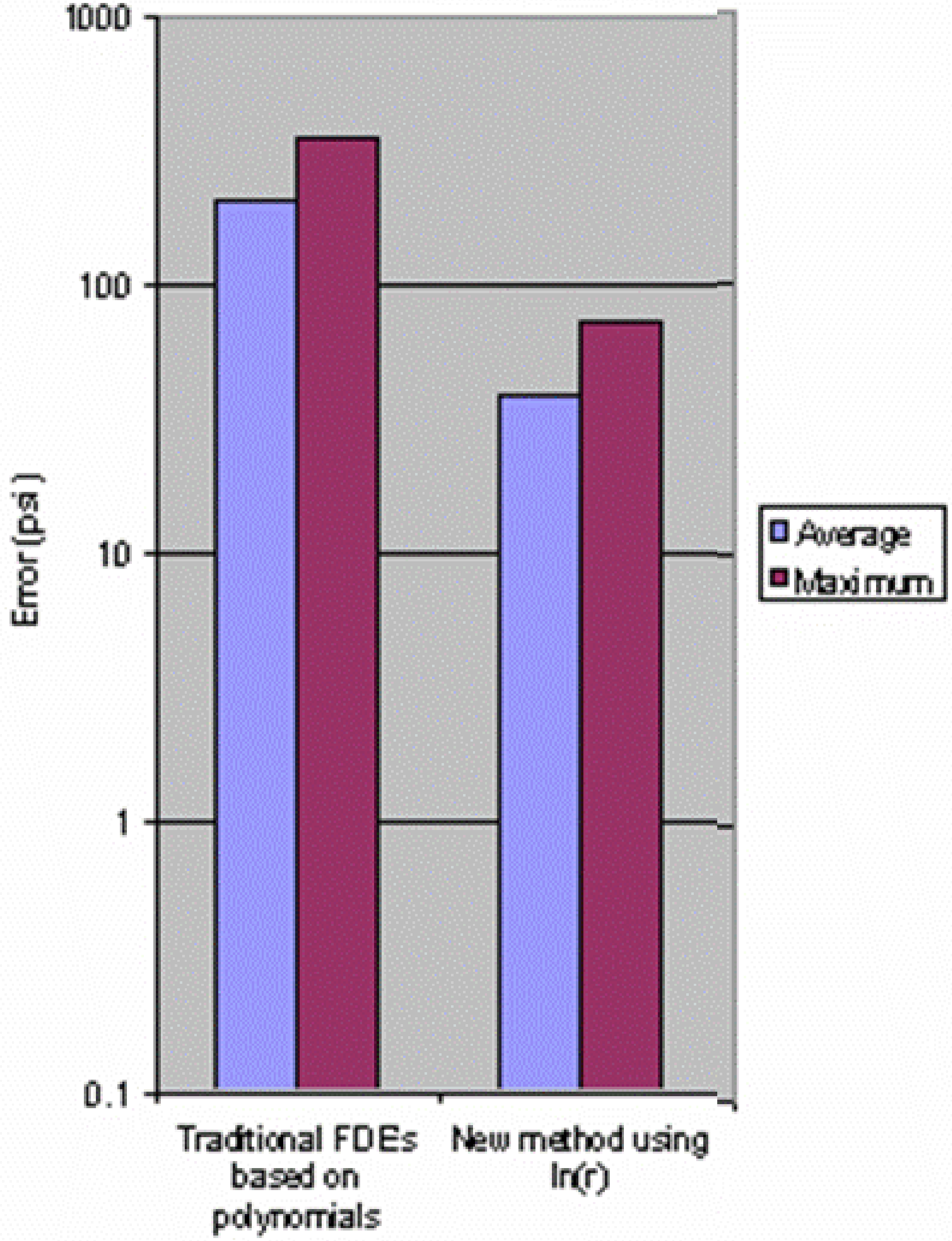
Hypothetical Reservoir

Two Line Source in rectangular reservoir



(Exact solution from Morel-Seytoux)

Results

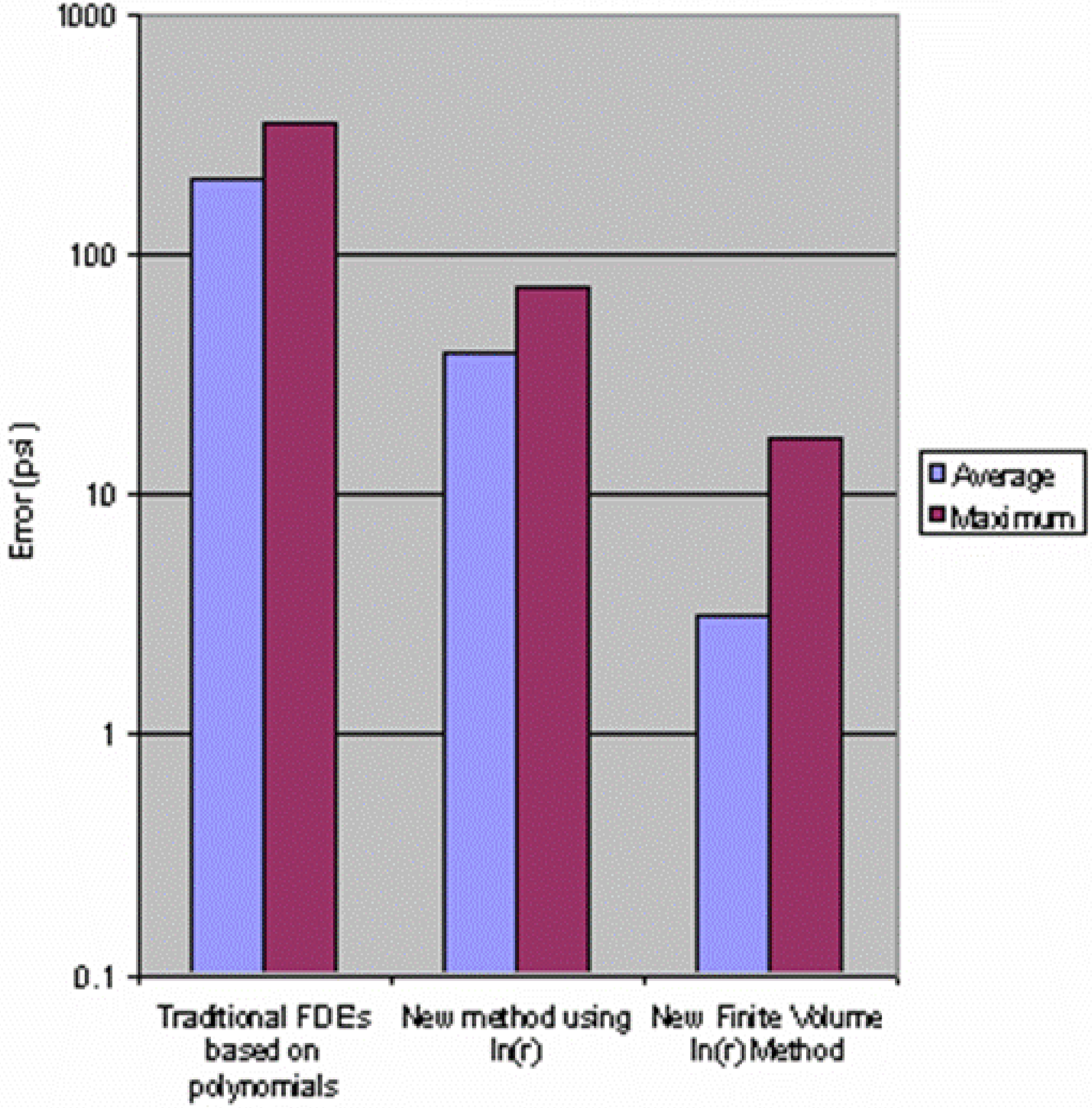


1000
100
10
1
0.1
Error (psi)
Average
Maximum
Traditional FDEs based on polynomials
New method using ln(r)

Finite Volume Equations

- Integrate over cell face to define the average flux
- Use average value instead of value at one point
- Result in a more accurate model

Results



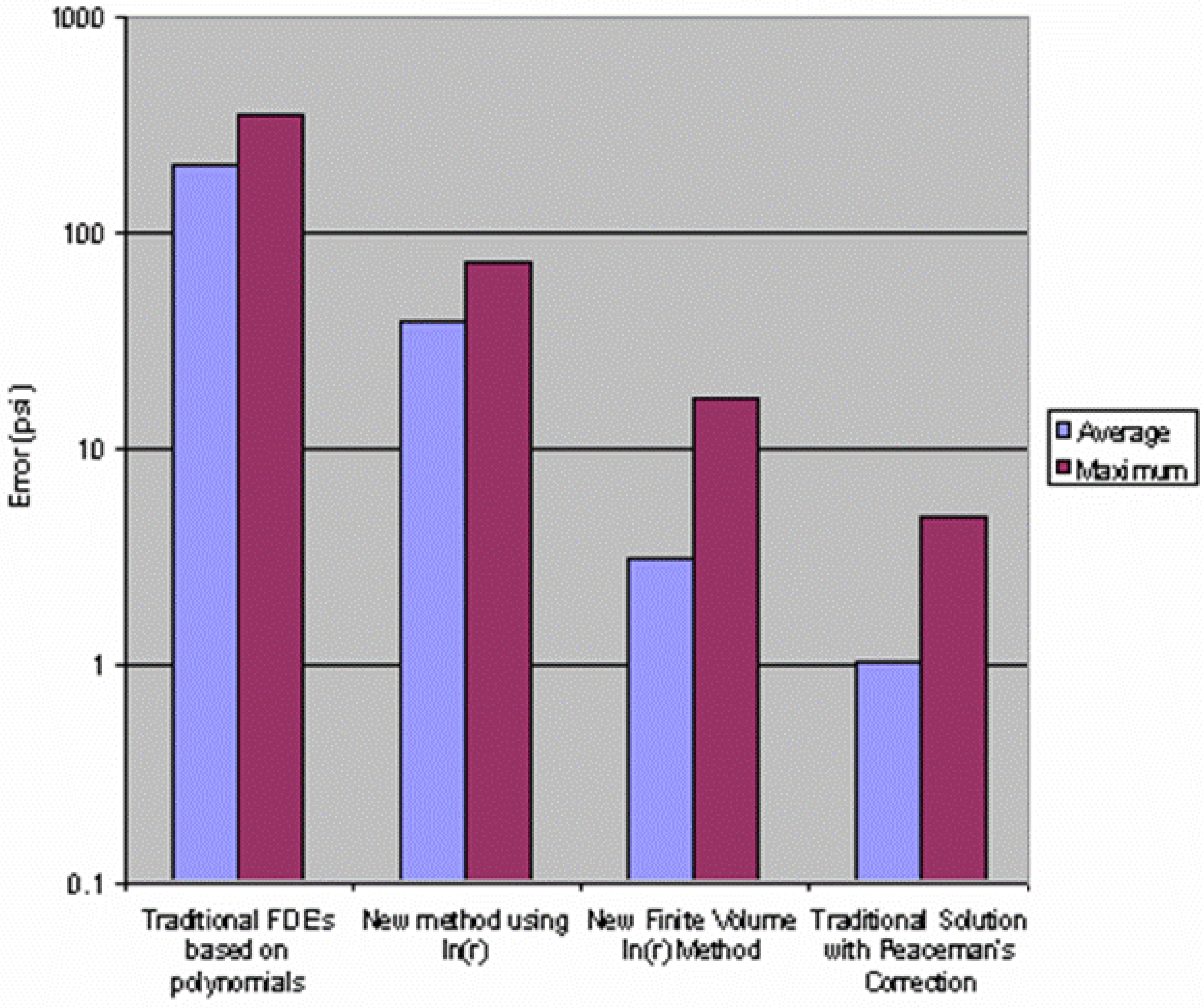
1000
100
10
1
0.1
Error (psi)
Traditional FDEs based on polynomials
New method using ln(r)
New Finite Volume ln(r) Method
Average
Maximum

Peaceman's Corrections

Peaceman's 1978 Well Equation

$$p_{well} - p_{cell} = \frac{1}{2\pi}\frac{q\mu}{Kh}\ln\left(\frac{c\Delta x}{r_w}\right) \quad \text{where } c \cong 0.2$$

Results

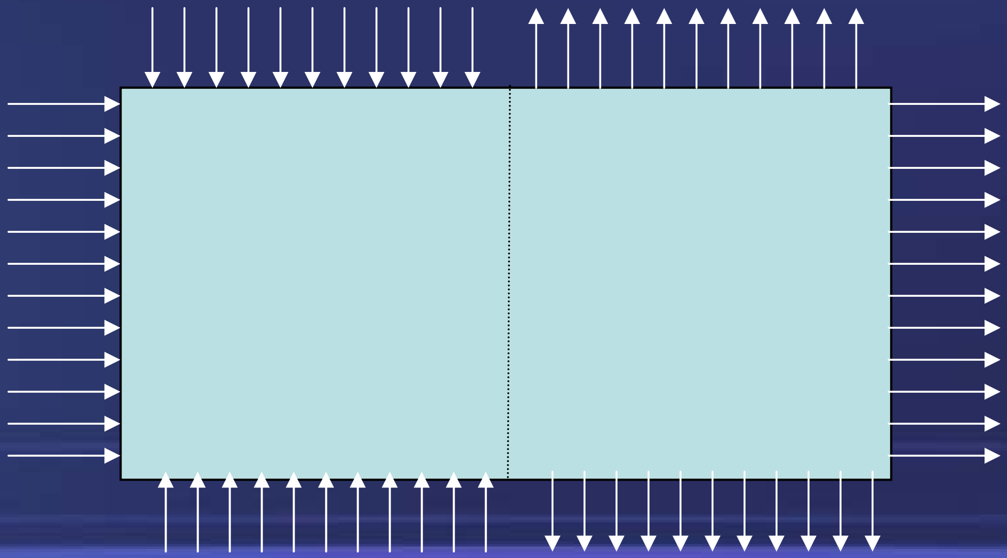


1000
100
10
1
0.1
Error (psi)
Traditional FDEs based on polynomials
New method using ln(r)
New Finite Volume ln(r) Method
Traditional Solution with Peaceman's Correction
Average
Maximum

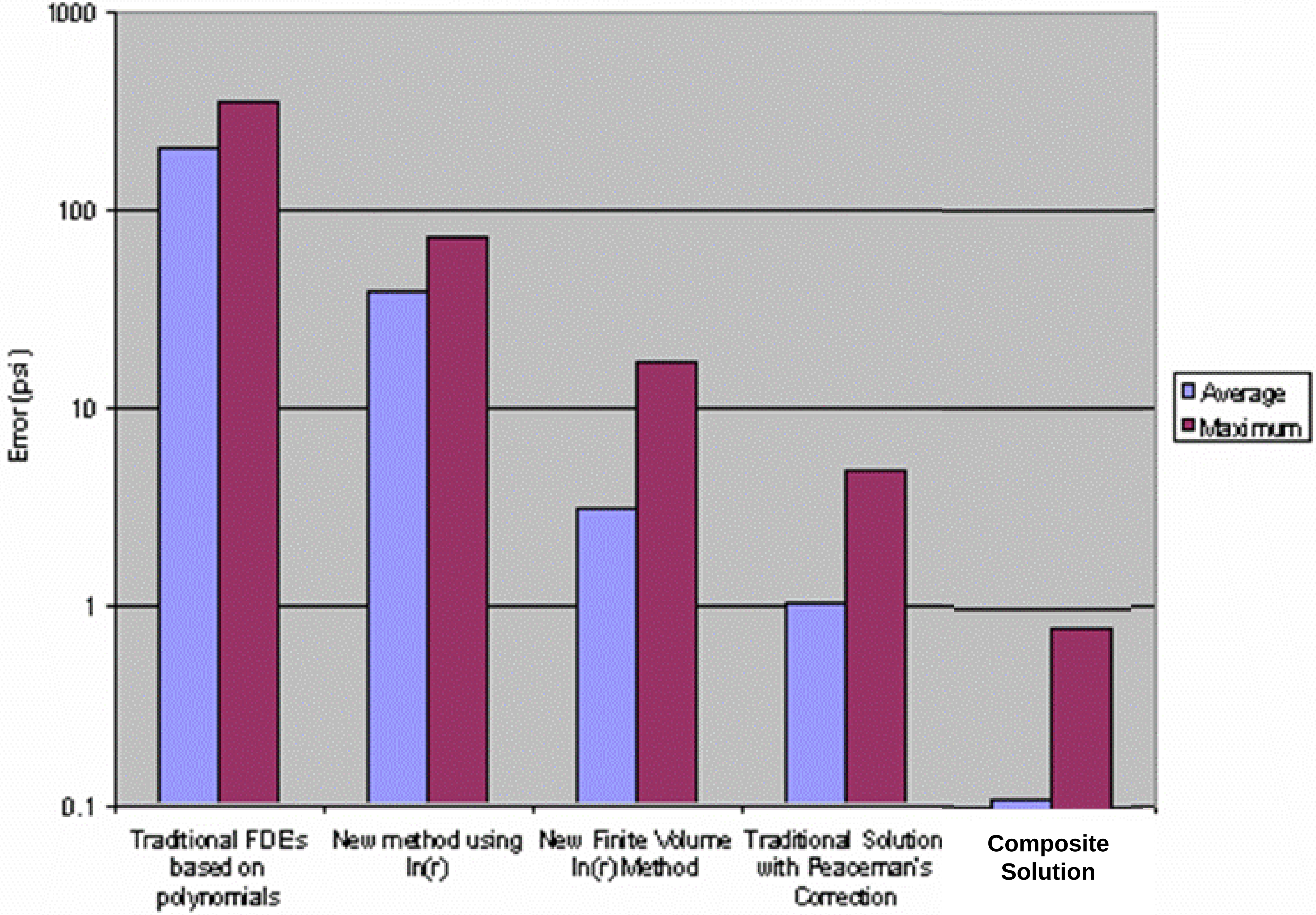
Composite Solution based on:

$$P = P_w + \frac{Q\mu}{2\pi Kh}\ln\left(\frac{r}{r_w}\right) + ax^2 + bx + c$$

$$P = P_a + P_f$$



Results



1000
100
10
1
0.1
Error (psi)
Average
Maximum
Traditional FDEs based on polynomials
New method using ln(r)
New Finite Volume ln(r) Method
Traditional Solution with Peaceman's Correction
Composite Solution

Conclusions

- Finite difference equations based on equations that include ln(r) terms improve reservoir simulation results considerably.
- Finite difference equations for the simulation of other processes may enjoy similar improvements through the incorporation of FDE's based on approximate physical solutions.